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Quantum symmetries and the Weyl–Wigner product of group representations

A J Bracken[1,2], **G Cassinelli**[1] **and J G Wood**[2]

[1] DIFI, Università di Genova, Via Dodecaneso 33, Genova 16146, Italy
[2] Centre for Mathematical Physics, Department of Mathematics, The University of Queensland, Brisbane 4072, Australia

E-mail: ajb@maths.uq.edu.au, cassinelli@ge.infn.it and jgw@maths.uq.edu.au

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Abstract

In the usual formulation of quantum mechanics, groups of automorphisms of quantum states have ray representations by unitary and antiunitary operators on complex Hilbert space, in accordance with Wigner's theorem. In the phase-space formulation, they have real, true unitary representations in the space of square-integrable functions on phase space. Each such phase-space representation is a Weyl–Wigner product of the corresponding Hilbert space representation with its contragredient, and these can be recovered by 'factorizing' the Weyl–Wigner product. However, not every real, unitary representation on phase space corresponds to a group of automorphisms, so not every such representation is in the form of a Weyl–Wigner product and can be factorized. The conditions under which this is possible are examined. Examples are presented.

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1. Introduction

Since the pioneering works of Weyl [1], von Neumann [2], Wigner [3], Groenewold [4] and Moyal [5], the phase-space formulation of quantum mechanics has been the subject of much research from many different points of view. The underlying theory has been greatly developed [6–27], including group-theoretical aspects of particular relevance to the present work [28–45].

Our interest here is in the way that symmetries, and more generally, groups of automorphisms of quantum states, are expressed by group representations in the formulation of quantum mechanics on phase space Γ, and the relationship of these to more familiar representations on complex Hilbert space $\mathcal{H}$. Representations of automorphism groups on Γ are typically true, real, unitary representations, whereas representations on $\mathcal{H}$ can be projective, and even antiunitary (such as in the case of time-reversal symmetry), in accordance with Wigner's theorem [46–48]. Such a phase-space representation Π_Γ is isomorphic (but not equal) [37] to the tensor product of a corresponding Hilbert space representation $\Pi_{\mathcal{H}}$ with its contragredient $\Pi^C_{\mathcal{H}}$. We call it the Weyl–Wigner product of $\Pi_{\mathcal{H}}$ and $\Pi^C_{\mathcal{H}}$, and write

$$\Pi_\Gamma = \Pi_{\mathcal{H}} \overset{W}{\otimes} \Pi^C_{\mathcal{H}} \cong \Pi_{\mathcal{H}} \otimes \Pi^C_{\mathcal{H}}. \tag{1}$$

Recent successes of 'quantum tomography' [49] have highlighted the fact that the quantum state vector (wavefunction) in $\mathcal{H}$ can be recovered from the Wigner distribution function on Γ, up to a constant phase [16]. In principle, the whole Hilbert space structure of quantum mechanics can be recovered from the phase-space structure [25], so we must expect that a projective, complex, unitary or antiunitary representation $\Pi_{\mathcal{H}}$ in Hilbert space can be recovered from the corresponding true, real, unitary representation Π_Γ in phase space, in effect by 'factorizing' Π_Γ as a Weyl–Wigner product (1). We shall confirm that this is the case. It is remarkable that this is possible, in particular because ray representations are associated with central extensions at the Lie algebra level, and it can only happen if the associated extension parameters (mass of a particle, Planck's constant, etc) already appear in the true, phase-space representation, or else arise in the mapping from phase space back to Hilbert space. We shall see that both possibilities are realized.

The structure of the phase-space formulation in its original form is intimately connected with the structure of the Heisenberg–Weyl group. Extensions to other groups have been described [7, 36, 43], but we shall deal here only with the original form, restricting Γ to the phase plane coordinatized by the pair (q, p). However, we shall be concerned with representations on Γ and $\mathcal{H}$ of groups and Lie algebras other than the Heisenberg–Weyl group and algebra. Generalizations to quantum systems with several degrees of freedom, and systems with spin, are certainly possible.

At the heart of the phase-space formulation of quantum mechanics lies the Weyl–Wigner transform $\mathcal{W}$, which is an invertible mapping from linear operators $\hat{A}, \hat{B}, \ldots$ on $\mathcal{H}$ to functions $A, B, \ldots$ on Γ.

Before embarking on a discussion of automorphism groups and their representations, it is necessary to outline a firm mathematical basis for the Weyl–Wigner transform and its inverse. More detail can be found in the literature [12, 25, 27]. We work with dimensionless variables in what follows, in effect setting Planck's constant $\hbar$ equal to 1, except in the first two examples at the end of the paper.

2. Background: a mathematical setting for the Weyl–Wigner transform

For our purposes, an appropriate setting for a description of $\mathcal{W}$ and $\mathcal{W}^{-1}$ for a system with one degree of freedom involves [31, 12, 25]

- the complex vector space of Hilbert–Schmidt operators on $\mathcal{H}$, regarded as a Hilbert space $\mathcal{T}_C$ with scalar product

$$(\hat{A}, \hat{B})_{\mathcal{T}_C} = \mathrm{Tr}(\hat{A}^\dagger \hat{B}) \tag{2}$$

- the complex vector space $L_2(\mathbb{C}, \mathrm{d}\Gamma)$, regarded as a Hilbert space $\mathcal{K}_C$ with scalar product

$$(A, B)_{\mathcal{K}_C} = \frac{1}{2\pi}\int \overline{A} B \,\mathrm{d}\Gamma \qquad \mathrm{d}\Gamma = \mathrm{d}q\,\mathrm{d}p \tag{3}$$

together with certain associated vector spaces. (We use Tr to denote the trace, and the overbar to denote complex conjugation. Integrals are over all real values of the variables of integration, unless otherwise indicated.)

The Hilbert space $\mathcal{H}$ of state vectors can be realized as $L_2(\mathbb{C}, \mathrm{d}x)$ (the 'coordinate representation') with scalar product

$$(\varphi, \psi)_{\mathcal{H}} = \int \overline{\varphi(x)}\psi(x)\,\mathrm{d}x. \tag{4}$$

Let $e_1, e_2, \ldots$ form an orthonormal basis of 'test' functions in this realization of $\mathcal{H}$. Each e_r and its Fourier transform is infinitely differentiable and each, together with all its derivatives, vanishes more quickly than any negative power of its argument, as that argument approaches $\pm\infty$; the eigenfunctions of the Hamiltonian operator of a simple harmonic oscillator provide an example. Introduce the 'Gel'fand triple' of vector spaces

$$\mathcal{G} < \mathcal{H} < \mathcal{G}' \tag{5}$$

where $\mathcal{G}$ is the Schwartz space associated with the basis $\{e_r\}$, and $\mathcal{G}'$ is its strong dual [51–54, 25].

Now let $\hat{e}_{rs}$, for $r, s = 1, 2, \ldots$ denote the rank-1 operator on $\mathcal{H}$ corresponding to the above choice of basis, defined by

$$\hat{e}_{rs}\varphi = (e_s, \varphi)_{\mathcal{H}} e_r \qquad \forall\varphi \in \mathcal{H}. \tag{6}$$

Then the set of $\hat{e}_{rs}$ forms an orthonormal basis in $\mathcal{T}_C$, with

$$(\hat{e}_{rs}, \hat{e}_{uv})_{\mathcal{T}_C} = \delta_{ru}\delta_{sv}. \tag{7}$$

Introduce the Gel'fand triple

$$\mathcal{S}_C < \mathcal{T}_C < \mathcal{S}'_C \tag{8}$$

by analogy with (5).

Corresponding to each $\hat{e}_{rs} \in \mathcal{T}_C$, define $\Phi_{rs} \in \mathcal{K}_C$ by

$$\Phi_{rs}(q, p) = \int e_r(q - y/2)\overline{e_s(q + y/2)}\,\mathrm{e}^{\mathrm{i}py}\,\mathrm{d}y. \tag{9}$$

It is easily checked that the set of Φ_{rs} forms an orthonormal basis in $\mathcal{K}_C$, and that each Φ_{rs} is a 'test function' of two variables. Introduce the Gel'fand triple

$$\mathcal{J}_C < \mathcal{K}_C < \mathcal{J}'_C \tag{10}$$

by analogy with (5) and (8).

The elements of $\mathcal{G}'$ are 'generalized functions' on the real line. Similarly, $\mathcal{J}'_C$ consists of generalized functions on the phase plane. The elements of $\mathcal{S}'_C$ are 'generalized linear operators', and include the operators in $\mathcal{S}_C$ and $\mathcal{T}_C$. It is not difficult to see that a generalized linear operator can be regarded as carrying elements of $\mathcal{G}$ into elements of $\mathcal{G}'$ in general, that is to say, test functions of one variable into generalized functions of one variable [25].

The Weyl–Wigner transform is a 1-1 invertible mapping from $\mathcal{S}'_C$ onto $\mathcal{J}'_C$ which associates a generalized function A with each generalized operator $\hat{A}$. Note first that each $\hat{A} \in \mathcal{S}_C$ can be interpreted as an integral operator

$$(\hat{A}\varphi)(x) = \int A_K(x, y)\varphi(y)\,\mathrm{d}y \qquad \varphi \in \mathcal{H} \tag{11}$$

whose kernel A_K is a test function of two variables. Then define

$$A(q, p) = (\mathcal{W}(\hat{A}))(q, p) = \int A_K(q - y/2, q + y/2)\,\mathrm{e}^{\mathrm{i}py}\,\mathrm{d}y \tag{12}$$

with inverse

$$A_K(x, y) = (\mathcal{W}^{-1}(A))_K(x, y) = \frac{1}{2\pi}\int A((x + y)/2, p)\,\mathrm{e}^{\mathrm{i}p(x-y)}\,\mathrm{d}p. \tag{13}$$

These formulae (12) and (13) can be extended to apply to every operator in $\mathcal{T}_C$ (regarded as an integral operator) and every function in $\mathcal{K}_C$ if the integrals are interpreted in the usual generalized way for Fourier transforms of L_2 functions.

Once $\mathcal{W}$ and $\mathcal{W}^{-1}$ have been defined in this way on $\mathcal{S}_C$ and $\mathcal{J}_C$, respectively, their definitions can be extended easily to $\mathcal{S}'_C$ and $\mathcal{J}'_C$, respectively, as follows. For each $\tau \in \mathcal{S}'_C$, define $\mathcal{W}(\tau) \in \mathcal{J}'_C$ by

$$\mathcal{W}(\tau)(\kappa) = \tau(\mathcal{W}^{-1}(\kappa)) \qquad \forall \kappa \in \mathcal{J}_C \tag{14}$$

and conversely, for each $\kappa \in \mathcal{J}'_C$, define $\mathcal{W}^{-1}(\kappa) \in \mathcal{S}'_C$ by

$$\mathcal{W}^{-1}(\kappa)(\tau) = \kappa(\mathcal{W}(\tau)) \qquad \forall \tau \in \mathcal{S}_C. \tag{15}$$

This defines $\mathcal{W}$ and $\mathcal{W}^{-1}$ as mappings from $\mathcal{S}'_C$ onto $\mathcal{J}'_C$ and vice versa. The mappings are continuous in the natural topologies on these spaces [25].

In particular, $\mathcal{W}$ and $\mathcal{W}^{-1}$ map $\mathcal{T}_C$ onto $\mathcal{K}_C$ and vice versa. In this case, as can be seen from (12) and (13), we have for every $\hat{A}, \hat{B} \in \mathcal{T}_C$ and corresponding $A, B \in \mathcal{K}_C$,

$$(A, B)_{\mathcal{K}_C} = (\hat{A}, \hat{B})_{\mathcal{T}_C} \tag{16}$$

showing that $\mathcal{W}$ and $\mathcal{W}^{-1}$ act as unitary transformations from $\mathcal{T}_C$ onto $\mathcal{K}_C$ and vice versa.

We note that $\mathcal{S}'_C$ contains two important classes of operators with the property that every operator in each class has every $\psi \in \mathcal{G}$ in its domain:

- The class of Hilbert–Schmidt operators, forming $\mathcal{T}_C$, which are bounded and defined on all of $\mathcal{H}$.
- The class Q of operators which leave $\mathcal{G}$ invariant, and so have $\mathcal{G}$ as a common, invariant domain, dense in $\mathcal{H}$. This class contains in particular the unit operator $\hat{I}$ on $\mathcal{H}$ and the canonical operators $\hat{q}, \hat{p}$ defined on $\psi \in \mathcal{G}$ by

$$\hat{q}\psi(x) = x\psi(x) \qquad \hat{p}\psi(x) = -\mathrm{i}\psi'(x) \tag{17}$$

  and it therefore also contains all polynomials in these operators, forming a subclass $Q_{WH} \subset Q$. We can say that Q_{WH} defines a representation on $\mathcal{G}$ of the enveloping algebra of the Heisenberg–Weyl Lie algebra.

The classes $\mathcal{T}_C$, Q and Q_{WH} share another important property: each is invariant under the formation of operator products. Note that $\mathcal{T}_C$ and Q are not disjoint, and that neither is a subclass of the other.

For $\hat{A}, \hat{B} \in \mathcal{T}_C$, we define the associative but noncommutative star product [4, 5] of the corresponding $A, B \in \mathcal{K}_C$ by

$$A \star B (= \mathcal{W}(\hat{A}) \star \mathcal{W}(\hat{B})) = \mathcal{W}(\hat{A}\hat{B}). \tag{18}$$

The Wigner transform defines not only a unitary transformation from $\mathcal{T}_C$ to $\mathcal{K}_C$, but also an isomorphism of these two sets, regarded as algebras. The usual operator product in $\mathcal{T}_C$ is replaced by the star product of functions in $\mathcal{K}_C$. The image of $(-\mathrm{i}\times)$ the commutator on $\mathcal{T}_C$ is the Groenewold–Moyal [4, 5] bracket on $\mathcal{K}_C$:

$$\{A, B\}_{GM} = -\mathrm{i}(A \star B - B \star A). \tag{19}$$

For sufficiently smooth A and B, in particular for $A, B \in \mathcal{J}_C$, it can be seen from (12) and (13) that

$$(A \star B)(q_1, p_1) = \frac{1}{\pi^2} \int A(q_2, p_2) B(q_3, p_3)\, \mathrm{e}^{-2\mathrm{i}[p_1(q_2-q_3)+p_2(q_3-q_1)+p_3(q_1-q_2)]}\, \mathrm{d}q_2\, \mathrm{d}p_2\, \mathrm{d}q_3\, \mathrm{d}p_3 \tag{20}$$

and so

$$\{A, B\}_{GM}(q_1, p_1) = -\frac{2}{\pi^2}\int A(q_2, p_2)B(q_3, p_3)\sin(2[p_1(q_2-q_3) + p_2(q_3-q_1) + p_3(q_1-q_2)])\,\mathrm{d}q_2\,\mathrm{d}p_2\,\mathrm{d}q_3\,\mathrm{d}p_3. \tag{21}$$

For such A and B, the order of the integrations is unimportant. For general $A, B \in \mathcal{K}_C$, (20) and (21) are valid with a generalized interpretation of the integrals.

The image under $\mathcal{W}$ of the space Q_{WH} is the subspace $\mathcal{I}_{WH}$ of $\mathcal{J}'_C$, consisting of polynomials in $1, q$ and p. In particular, $\mathcal{W}(\hat{I}) = 1, \mathcal{W}(\hat{q}) = q$ and $\mathcal{W}(\hat{p}) = p$. For $\hat{A}, \hat{B} \in Q_{WH}$, we again use (18) and (19) to define the star product and Groenewold–Moyal bracket of the corresponding $A, B \in \mathcal{I}_{WH}$. The transforms $\mathcal{W}$ and $\mathcal{W}^{-1}$ map $\mathcal{Q}_{WH}$ onto $\mathcal{I}_{WH}$ and vice versa, preserving polynomial degree. This action establishes an equivalence of two representations of the enveloping algebra of the Heisenberg–Weyl Lie algebra, one in $\mathcal{Q}_{WH}$ with the usual operator product, the other in $\mathcal{I}_{WH}$ with the star product. The structure of the mappings $\mathcal{W}$ and $\mathcal{W}^{-1}$ in this case is well known [1, 10, 11, 16]. We have

$$\mathcal{W}(\hat{q}^n\hat{p}^m) = \sum_{k=0}^{\min(m,n)} \left(\frac{\mathrm{i}}{2}\right)^k k!C_k^m C_k^n q^{m-k}p^{n-k} \tag{22}$$

where $C_r^m = m!/(r!(m-r)!)$. Conversely,

$$\mathcal{W}^{-1}(q^m p^n) = \sum_{k=0}^{\min(m,n)} \left(\frac{-\mathrm{i}}{2}\right)^k k!C_k^m C_k^n \hat{q}^{m-k}\hat{p}^{n-k} = \frac{1}{2^m}\sum_{r=0}^{m} C_r^m \hat{q}^{m-r}\hat{p}^n\hat{q}^r. \tag{23}$$

Further similar formulae can be obtained by replacing $\hat{q}$ by $\hat{p}$, $\hat{p}$ by $-\hat{q}$, q by p and p by $-q$.

On $\mathcal{I}_{WH}$, the star product reduces to

$$(A \star B)(q, p) = A(q, p)B(q, p) + \mathrm{i}(AJB)(q, p) - \frac{1}{2!}(AJ^2B)(q, p) + \cdots = B(q, p)A(q, p) - \mathrm{i}(BJA)(q, p) - \frac{1}{2!}(BJ^2A)(q, p) + \cdots \tag{24}$$

where

$$J = \frac{1}{2}\left(\frac{\partial^{(L)}}{\partial q}\frac{\partial^{(R)}}{\partial p} - \frac{\partial^{(R)}}{\partial q}\frac{\partial^{(L)}}{\partial p}\right) \tag{25}$$

with L and R indicating the directions in which the differential operators act. Then the Groenewold–Moyal bracket (19) takes the form

$$(\{A, B\}_{GM})(q, p) = 2((AJB)(q, p) - \frac{1}{3!}(AJ^3B)(q, p) + \frac{1}{5!}(AJ^5B)(q, p) + \cdots). \tag{26}$$

The formulae (24) and (26) are commonly written as

$$A \star B = A\mathrm{e}^{\mathrm{i}J}B = B\mathrm{e}^{-\mathrm{i}J}A \qquad \{A, B\}_{GM} = 2A\sin(J)B. \tag{27}$$

Note however that because A and B in Q_{WH} are polynomials, the series in (24) and (26) terminate.

More generally, we can use (18) and (19) to define the star product and Groenewold–Moyal bracket of those $A, B \in \mathcal{J}'_C$ corresponding to $\hat{A}, \hat{B} \in Q$. Note that, for all $\hat{A} \in \mathcal{S}'_C$,

$$\mathcal{W}(\hat{A}) = A \Leftrightarrow \mathcal{W}(\hat{A}^\dagger) = \overline{A} \tag{28}$$

and that whenever the star product of A and B is defined, it satisfies

$$\overline{A \star B} = \overline{B} \star \overline{A} \tag{29}$$

reflecting the fact that

$$(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger \hat{A}^\dagger. \tag{30}$$

3. Quantum states

Let $\mathcal{T}_R$ denote the Hilbert space of self-adjoint Hilbert–Schmidt operators over the real numbers, with scalar product

$$(\hat{A}, \hat{B})_{\mathcal{T}_R} = \mathrm{Tr}(\hat{A}\hat{B}). \tag{31}$$

Its image under $\mathcal{W}$ is $\mathcal{K}_R$, the Hilbert space of square-integrable, real-valued functions on Γ with scalar product

$$(A, B)_{\mathcal{K}_R} = \frac{1}{2\pi}\int AB\,\mathrm{d}\Gamma. \tag{32}$$

The elements of $\mathcal{T}_R$ and $\mathcal{K}_R$ represent a class of observables on a quantum system, in the Hilbert space and phase-space formulations, respectively. By an obvious extension of the arguments for $\mathcal{T}_C$ and $\mathcal{K}_C$, the mappings $\mathcal{W}$ and $\mathcal{W}^{-1}$ act as unitary transformations between $\mathcal{T}_R$ and $\mathcal{K}_R$: if $A = \mathcal{W}(\hat{A})$ and $B = \mathcal{W}(\hat{B})$, we have

$$(\hat{A}, \hat{B})_{\mathcal{T}_R} = (A, B)_{\mathcal{K}_R}. \tag{33}$$

Let $\mathbf{P} \subset \mathcal{T}_R$ denote the set of pure state density operators $\hat{\rho} \in \mathcal{T}_R$ for the quantum system, which are characterized by the conditions

$$\hat{\rho}^2 = \hat{\rho} \qquad (\hat{\rho}, \hat{\rho})_{\mathcal{T}_R} = 1. \tag{34}$$

The set of pure and mixed state density operators is the convex set of $\hat{\rho} \in \mathcal{T}_R$ with the pure state density operators as extremal points. Corresponding to each $\hat{\rho}$, pure or mixed, the Wigner distribution function is defined as

$$W = \frac{1}{2\pi}\mathcal{W}(\hat{\rho}). \tag{35}$$

It follows at once from (31)–(33) that

$$\mathrm{Tr}(\hat{\rho}\hat{A}) = \int W(q, p)A(q, p)\,\mathrm{d}\Gamma \tag{36}$$

for each $\hat{A} \in \mathcal{T}_R$ and corresponding $A \in \mathcal{K}_R$, which equates the familiar expressions for quantum averages in the Hilbert space and phase-space forms.

Let $\mathbf{V} = \mathcal{W}(\mathbf{P}) \subset \mathcal{K}_R$ denote the set of pure state Wigner functions. For any $W \in \mathbf{V}$ we have, from (35) and (34),

$$W \star W = \frac{1}{2\pi}W \qquad 2\pi\int W^2\,\mathrm{d}\Gamma = \int W\,\mathrm{d}\Gamma = 1. \tag{37}$$

Pure state density operators, and hence pure state Wigner functions, are in one-to-one correspondence with unit rays in the Hilbert space of state vectors.

Given a unit ray, the corresponding pure state density operator is the one-dimensional projection whose action on any $\chi \in \mathcal{H}$ is given by

$$\hat{\rho}\chi = (\psi, \chi)_{\mathcal{H}}\psi \tag{38}$$

where ψ is any vector in the ray. In the coordinate representation adopted in section 2, $\hat{\rho}$ is the integral operator with kernel $\psi(x)\overline{\psi(y)}$, and the corresponding Wigner function takes the form, from (12),

$$W(q,p) = \frac{1}{2\pi}\int \psi(q-y/2)\overline{\psi(q+y/2)}\,\mathrm{e}^{\mathrm{i}py}\,\mathrm{d}y. \tag{39}$$

The inverse problem, of finding the unit ray corresponding to a given pure state Wigner function W satisfying (37), which is equivalent to the problem of finding the unit ray corresponding to a given pure state density operator $\hat{\rho}$ satisfying (34), has been treated by Tatarskii [16]. However, it is difficult to define a linear mapping from Wigner functions to corresponding wavefunctions. This is an obstacle to recovering a (linear) unitary mapping between wavefunctions in $\mathcal{H}$, corresponding to a given mapping between Wigner functions, associated with a transformation from some symmetry group, say. On the other hand, it is known that the Hilbert space structure is represented within the phase-space structure [25]. We shall see that it is possible in principle to recover such unitary symmetry operators on $\mathcal{H}$, as well as antiunitary symmetry operators, using a different approach.

4. Automorphisms and Wigner's theorem

Let Aut(**P**) denote the set of automorphisms of **P**. It consists of all bijective maps $\mu : \mathbf{P} \to \mathbf{P}$ that also satisfy the condition

$$(\mu(\hat{\rho}_1), \mu(\hat{\rho}_2))_{\mathcal{T}_R} = (\hat{\rho}_1, \hat{\rho}_2)_{\mathcal{T}_R} \tag{40}$$

for all $\hat{\rho}_1, \hat{\rho}_2 \in \mathbf{P}$, and is a group under the natural composition of mappings. We refer to the mappings in Aut(**P**) as **P**-*automorphisms*.

Let Aut(**V**) denote the set of automorphisms of **V**. It consists of all bijective maps $M : \mathbf{V} \to \mathbf{V}$ that also satisfy the condition

$$(M(W_1), M(W_2))_{\mathcal{K}_R} = (W_1, W_2)_{\mathcal{K}_R} \tag{41}$$

for all $W_1, W_2 \in \mathbf{V}$, and similarly forms a group. We refer to the mappings in Aut(**V**) as **V**-*automorphisms*.

The Weyl–Wigner transform defines a unitary transformation from $\mathcal{T}_R$ to $\mathcal{K}_R$ which maps **P** onto **V**, and establishes an isomorphism of Aut(**P**) and Aut(**V**). Explicitly,

$$M(W) = M(\mathcal{W}(\hat{\rho})) = \mathcal{W}(\mu(\hat{\rho})) \qquad \mu(\hat{\rho}) = \mu(\mathcal{W}^{-1}(W)) = \mathcal{W}^{-1}(M(W)). \tag{42}$$

According to Wigner's theorem [46–48], given any $\mu \in$ Aut(**P**), there exists a unitary or antiunitary operator $\hat{U}$ on $\mathcal{H}$, unique up to a phase factor, such that

$$\mu(\hat{\rho}) = \hat{U}\hat{\rho}\hat{U}^{\dagger} \qquad \forall \hat{\rho} \in \mathbf{P}. \tag{43}$$

Each $\mu \in$ Aut(**P**) extends to an operator on $\mathcal{T}_C$, with

$$\mu(\hat{A}) = \hat{U}\hat{A}\hat{U}^{\dagger} \qquad \forall \hat{A} \in \mathcal{T}_C. \tag{44}$$

This operator does not act linearly on $\mathcal{T}_C$ in general, but it does always act linearly on $\mathcal{T}_R$, which it leaves invariant. It defines a real unitary transformation of $\mathcal{T}_R$ onto itself. We denote this transformation also by μ. We extend the whole group of automorphisms Aut(**P**) in this way to act on all of $\mathcal{T}_R$, and denote this group with extended domain of action also by Aut(**P**). Similarly, we extend each automorphism $M \in$ Aut(**V**), and hence Aut(**V**) itself, to act unitarily on all of $\mathcal{K}_R$.

Aut(**P**) is isomorphic to $\Sigma(\mathcal{H})$, the group of unitary and antiunitary operators on $\mathcal{H}$, factored by its closed centre, the phase group [48]:

$$\Sigma(\mathcal{H}) = \mathbf{U} \cup \overline{\mathbf{U}}/\mathbf{T} \tag{45}$$

and it follows that this is also true of Aut(**V**).

In the case that $\hat{U}$ is unitary, the action of M on $A \in \mathcal{K}_R$ corresponding to (44) is given by

$$M(A) = \mathcal{W}(\hat{U}\hat{A}\hat{U}^\dagger) = \mathcal{W}(\hat{U}) \star A \star \mathcal{W}(\hat{U}^\dagger) = U \star A \star \overline{U}. \tag{46}$$

Here $U = \mathcal{W}(\hat{U})$ is a complex-valued function on Γ, and $\overline{U}$ is its complex conjugate. Corresponding to unitarity of $\hat{U}$ we have

$$U \star \overline{U} = \overline{U} \star U = 1. \tag{47}$$

Note that a unitary operator $\hat{U}$ lies in $\mathcal{S}'_C$, not in $\mathcal{T}_C$, but

$$\hat{A} \in \mathcal{T}_R \Rightarrow \hat{U}\hat{A}\hat{U}^\dagger \in \mathcal{T}_R. \tag{48}$$

Similarly, a star-unitary function U lies in $\mathcal{J}'_C$, not $\mathcal{K}_C$, but

$$A \in \mathcal{K}_R \Rightarrow U \star A \star \overline{U} \in \mathcal{K}_R. \tag{49}$$

In the case that $\hat{U}$ is antiunitary, because the action of $\mathcal{W}$ on antiunitary operators has not been defined, we proceed as follows. Consider the particular antiunitary operator $\hat{C}$ on $\mathcal{H}$ which leaves all basis vectors e_r invariant: if $\varphi = \sum_r \varphi_r e_r$, then

$$\hat{C}\varphi = \sum_r \overline{\varphi_r} e_r. \tag{50}$$

(If we work in the coordinate representation, and choose a basis in $\mathcal{H}$ of real-valued functions, then $\hat{C}$ is the operation of complex conjugation.)

Next, let $\mathcal{P}$ denote the operator on $\mathcal{K}_C$ defined by

$$(\mathcal{P}(A))(q, p) = A(q, -p) \qquad \text{for all} \quad A \in \mathcal{K}_C. \tag{51}$$

Then $\mathcal{P}$ is unitary on $\mathcal{K}_C$, and also (real) unitary when restricted to $\mathcal{K}_R$. It is evident that, on $\mathcal{K}_C$ or $\mathcal{K}_R$,

$$\mathcal{P}^\dagger = \mathcal{P} \qquad \mathcal{P}^2 = I. \tag{52}$$

Direct calculation from (20) shows also that

$$\mathcal{P}(A \star B) = \mathcal{P}(B) \star \mathcal{P}(A) \qquad \text{for all} \quad A, B \in \mathcal{K}_C. \tag{53}$$

The form of the Wigner function W' corresponding to $\hat{\rho}' = \hat{C}\hat{\rho}\hat{C}$ is now given from (39) by

$$W'(q, p) = W(q, -p) = (\mathcal{P}(W))(q, p) \tag{54}$$

and generalizing (54), we find for a general $\hat{A} \in \mathcal{T}_C$ and corresponding $A \in \mathcal{K}_C$, that

$$\mathcal{W}(\hat{C}\hat{A}\hat{C}) = \mathcal{P}(\overline{A}). \tag{55}$$

The transformation of W corresponding to a general antiunitary operator $\hat{U} = \hat{C}\hat{V}$ in (43), where $\hat{V}$ is unitary, is

$$\begin{aligned} &W' = \mathcal{P}(V \star W \star \overline{V}) = \mathcal{P}(\overline{V}) \star \mathcal{P}(W) \star \mathcal{P}(V) \qquad \text{or} \\ &W'(q, p) = (V \star W \star \overline{V})(q, -p) \end{aligned} \tag{56}$$

where $V = \mathcal{W}(\hat{V})$ satisfies the star-unitarity condition (47). More generally, for any $\hat{A} \in \mathcal{T}_C$ and corresponding $A \in \mathcal{K}_C$,

$$\mathcal{W}(\hat{U}\hat{A}\hat{U}^\dagger) = \mathcal{W}(\hat{C}\hat{V}\hat{A}\hat{V}^\dagger\hat{C}) = \mathcal{P}(V \star \overline{A} \star \overline{V}) = \mathcal{P}(\overline{V}) \star \mathcal{P}(\overline{A}) \star \mathcal{P}(V). \tag{57}$$

Let $O(\mathcal{T}_R)$ denote the group of all real unitary transformations of $\mathcal{T}_R$ onto itself, and let $O(\mathcal{K}_R)$ denote the group of real unitary transformations of $\mathcal{K}_R$ onto itself. Then

$$\begin{aligned} &\mathrm{Aut}(\mathbf{P}) \cong \mathrm{Aut}(\mathbf{V}) \qquad O(\mathcal{T}_R) \cong O(\mathcal{K}_R) \\ &\mathrm{Aut}(\mathbf{P}) < O(\mathcal{T}_R) \qquad \mathrm{Aut}(\mathbf{V}) < O(\mathcal{K}_R). \end{aligned} \tag{58}$$

In particular, it is important to note that in general Aut(**P**) and Aut(**V**) are proper subgroups of $O(\mathcal{T}_R)$ and $O(\mathcal{K}_R)$, respectively. It is easy to see that Aut(**P**) can be characterized as the subgroup of $O(\mathcal{T}_R)$ whose elements satisfy

$$\mu(\hat{A}\hat{B}) = \mu(\hat{A})\mu(\hat{B}) \qquad \forall \hat{A}, \hat{B} \in \mathcal{T}_R. \tag{59}$$

For if $\mu \in$ Aut(**P**), then (59) is satisfied as a consequence of (44), and conversely, if $\mu \in O(\mathcal{T}_R)$ satisfies (59), then it is a bijective map from **P** to **P** which satisfies (40), and so belongs to Aut(**P**). Likewise Aut(**V**) can be characterized as the subgroup of $O(\mathcal{K}_R)$ whose elements satisfy

$$M(A \star B) = M(A) \star M(B). \tag{60}$$

Given an element $\mu \in$ Aut(**P**) (or equivalently, given an element $M \in$ Aut(**V**)), it is possible in principle to construct the corresponding unitary or antiunitary operator $\hat{U}$ of (44), up to a phase, and proofs of Wigner's theorem show how it can be done [46–48]. However, there seems to be no simple recipe for such a construction in general. Fortunately, in many applications to physics, we have to deal with connected Lie groups of automorphisms, possibly extended by discrete transformations, and the problem of identifying the generator of a one-parameter group of unitaries corresponding to the generator of a given one-parameter group of automorphisms is more straightforward. This is exploited in what follows.

5. Symmetries and the Weyl–Wigner product

Given a group G and a quantum system having $\mathcal{H}$ as its space of state vectors, we say that G is a *pre-symmetry group* of the system, if there exists a homomorphism μ from G onto a subgroup $\bar{G} <$ Aut(**P**). The group of symmetries of the Hamiltonian of the system serves as one example, and any dynamical symmetry (or spectrum-generating) group as another, but Aut(**P**) is large, with many subgroups. Wigner's theorem [46–48] shows that Aut(**P**), and hence every pre-symmetry group G, has a ray representation $\Pi_{\mathcal{H}}$ by unitary and antiunitary operators on $\mathcal{H}$,

$$\Pi_{\mathcal{H}}(g)\varphi = \hat{U}(g)\varphi \qquad \hat{U}(g_1)\hat{U}(g_2) = \mathrm{e}^{\mathrm{i}\omega(g_1,g_2)}\hat{U}(g_1g_2) \tag{61}$$

where ω is a real-valued function satisfying appropriate associativity conditions [46, 47].

Of more direct interest to us here is that G has a real unitary representation $\Pi_{\mathcal{T}_R}$ on $\mathcal{T}_R$, and an isomorphic real unitary representation $\Pi_{\mathcal{K}_R}$ on $\mathcal{K}_R$. The representation $\Pi_{\mathcal{T}_R}$ is defined by the action (44) of each element $\mu(g) = \Pi_{\mathcal{T}_R}(g)$ of $\bar{G} <$ Aut(**P**) on an arbitrary element $\hat{A} \in \mathcal{T}_R$:

$$g : \hat{A} \longrightarrow \Pi_{\mathcal{T}_R}(g)(\hat{A}) = \hat{U}(g)\hat{A}\hat{U}(g)^{\dagger}. \tag{62}$$

The transformation $\Pi_{\mathcal{T}_R}(g)$ is real and unitary, even in the case that $\hat{U}(g)$ is antiunitary, as noted earlier. The group representation property is immediate from (62):

$$\begin{aligned}\Pi_{\mathcal{T}_R}(g_1)\Pi_{\mathcal{T}_R}(g_2)(\hat{A}) &= \hat{U}(g_1)\hat{U}(g_2)\hat{A}\hat{U}(g_2)^{\dagger}\hat{U}(g_1)^{\dagger}\\ &= \mathrm{e}^{\mathrm{i}\omega(g_1,g_2)}\hat{U}(g_1g_2)\hat{A}\hat{U}(g_1g_2)^{\dagger}\,\mathrm{e}^{-\mathrm{i}\omega(g_1,g_2)}\\ &= \Pi_{\mathcal{T}_R}(g_1g_2)(\hat{A}).\end{aligned} \tag{63}$$

The unitary representation $\Pi_{\mathcal{K}_R}$ is defined as the Weyl–Wigner transform of the unitary representation $\Pi_{\mathcal{T}_R}$, to which it is therefore isomorphic:

$$\begin{aligned}&\Pi_{\mathcal{K}_R}(g)(A) = \mathcal{W}\left(\Pi_{\mathcal{T}_R}(g)(\hat{A})\right) \qquad \text{for all} \quad \hat{A} \in \mathcal{T}_R\\ &\text{that is} \quad \Pi_{\mathcal{K}_R}\mathcal{W} = \mathcal{W}\Pi_{\mathcal{T}_R} \quad \text{on } \mathcal{T}_R.\end{aligned} \tag{64}$$

The group action of $\Pi_{\mathcal{K}_R}$ follows from that of $\Pi_{\mathcal{T}_R}$ in (63), but is worth considering in more detail. In the case that $\hat{U}(g)$ is unitary, the action of $\Pi_{\mathcal{K}_R}$ corresponding to (62) is

$$g : A \longrightarrow \Pi_{\mathcal{K}_R}(g)(A) = U(g) \star A \star \overline{U(g)} \tag{65}$$

where $U(g) = \mathcal{W}(\hat{U}(g))$ is star-unitary.

If every element $\hat{U}(g)$ of $\Pi_{\mathcal{H}}$ is unitary then, just as

$$\hat{U}(g_1)\hat{U}(g_2) = \mathrm{e}^{\mathrm{i}\omega(g_1,g_2)}\hat{U}(g_1g_2) \tag{66}$$

in (61), so the functions $U(g)$ satisfy

$$U(g_1) \star U(g_2) = \mathrm{e}^{\mathrm{i}\omega(g_1,g_2)}U(g_1g_2) \tag{67}$$

and provide a unitary ray representation under the star product, isomorphic to $\Pi_{\mathcal{H}}$. Such $\star$-representations have been discussed in the literature [31–34, 37, 27].

In the case that $\hat{U}(g) = \hat{C}\hat{V}(g)$ is antiunitary, with $\hat{V}(g)$ unitary and $\hat{C}$ the antiunitary operator in (50), the action of $\Pi_{\mathcal{K}_R}(g)$ is, corresponding to (62),

$$g : A \longrightarrow \Pi_{\mathcal{K}_R}(g)(A) = \mathcal{P}(V(g) \star A \star \overline{V(g)}) = \mathcal{P}(\overline{V(g)}) \star \mathcal{P}(A) \star \mathcal{P}(V(g)) \tag{68}$$

where $V(g) = \mathcal{W}(\hat{V}(g))$ is star-unitary. The group representation property for $\Pi_{\mathcal{K}_R}$, which is of course also guaranteed by the isomorphism between $\Pi_{\mathcal{T}_R}$ and $\Pi_{\mathcal{K}_R}$, can be regarded as a consequence of the star-unitarity of $\hat{U}(g)$ and $\hat{V}(g)$, and the properties (52) of $\mathcal{P}$. For example, if $\hat{U}(g_2)$ is unitary, but $\hat{U}(g_1) = \hat{C}\hat{V}(g_1)$ and $\hat{U}(g_1g_2) = \hat{C}\hat{V}(g_1g_2)$ are antiunitary, then

$$\Pi_{\mathcal{K}_R}(g_1)\Pi_{\mathcal{K}_R}(g_2)(A) \tag{69}$$

$$\begin{aligned}
&= \mathcal{P}(\overline{V(g_1)}) \star \mathcal{P}(\overline{U(g_2)}) \star \mathcal{P}(A) \star \mathcal{P}(U(g_2)) \star \mathcal{P}(V(g_1))\\
&= \mathcal{P}(\overline{U(g_2)} \star \overline{V(g_1)}) \star \mathcal{P}(A) \star \mathcal{P}(V(g_1) \star U(g_2))\\
&= \mathcal{P}(\overline{(V(g_1) \star U(g_2))}) \star \mathcal{P}(A) \star \mathcal{P}(V(g_1) \star U(g_2))\\
&= \mathcal{P}(\overline{V(g_1g_2)}) \star \mathcal{P}(A) \star \mathcal{P}(V(g_1g_2))\\
&= \mathcal{P}(V(g_1g_2) \star A \star \overline{V(g_1g_2)})\\
&= \Pi_{\mathcal{K}_R}(g_1g_2)(A).
\end{aligned} \tag{70}$$

The representation $\Pi_{\mathcal{T}_R}$ on $\mathcal{T}_R$, and hence the representation $\Pi_{\mathcal{K}_R}$ on $\mathcal{K}_R$, is isomorphic to the tensor product of the Hilbert space representation $\Pi_{\mathcal{H}}$ with its contragredient [37]:

$$\Pi_{\mathcal{T}_R} \cong \Pi_{\mathcal{K}_R} \cong \Pi_{\mathcal{H}} \otimes \Pi^C_{\mathcal{H}}. \tag{71}$$

To see this, we realize $\mathcal{H} \otimes \mathcal{H}$ as $L_2(\mathbb{C}, \mathrm{d}x) \otimes L_2(\mathbb{C}, \mathrm{d}y)$, then $\Pi_{\mathcal{H}}$ on $L_2(\mathbb{C}, \mathrm{d}x)$, and $\Pi^C_{\mathcal{H}}$ on $L_2(\mathbb{C}, \mathrm{d}y)$. Consider first the case that every element of $\Pi_{\mathcal{H}}$ is unitary. Let $\mathcal{N}$ denote the unitary mapping from $\mathcal{T}_R$ to $L_2(\mathbb{C}, \mathrm{d}x) \otimes L_2(\mathbb{C}, \mathrm{d}y)$ defined by

$$\mathcal{N}(\hat{A}) = A_K \tag{72}$$

where $A_K(x, y)$ is the kernel of $\hat{A}$, regarded as an integral operator, as in (11). Then

$$\mathcal{N}\left(\Pi_{\mathcal{T}_R}(g)(\hat{A})\right) = \int U_K(g|x, x')A_K(x', y')\overline{U_K(g|y', y)}\,\mathrm{d}x'\,\mathrm{d}y' \tag{73}$$

corresponding to (62). In (73), the kernel of $\hat{U}(g)$ is $U_K(g|x', y')$, which is not itself square-integrable. Because the action of $\Pi_{\mathcal{H}}(g)$ in $L_2(\mathbb{C}, \mathrm{d}x)$ is defined by

$$(\Pi_{\mathcal{H}}(g)\varphi)(x) = \int U_K(g|x, x')\varphi(x')\,\mathrm{d}x' \tag{74}$$

and the action of $\Pi^C_{\mathcal{H}}(g)$ in $L_2(\mathbb{C}, \mathrm{d}y)$ is defined by

$$\left(\Pi^C_{\mathcal{H}}(g)\varphi\right)(y) = \int \overline{U_K(g|y', y)}\varphi(y')\,\mathrm{d}y' \tag{75}$$

then (73) expresses the isomorphism between $\Pi_{\mathcal{T}_R}$ and $\Pi_{\mathcal{H}} \otimes \Pi^C_{\mathcal{H}}$:

$$\begin{aligned}\mathcal{N}\left(\Pi_{\mathcal{T}_R}(g)(\hat{A})\right) &= \left(\Pi_{L_2(\mathbb{C},\mathrm{d}x)}(g) \otimes \Pi^C_{L_2(\mathbb{C},\mathrm{d}y)}(g)\right)(\mathcal{N}(\hat{A})) \qquad \text{or} \\ \mathcal{N}\Pi_{\mathcal{T}_R} &= \left(\Pi_{L_2(\mathbb{C},\mathrm{d}x)}(g) \otimes \Pi^C_{L_2(\mathbb{C},\mathrm{d}y)}\right)\mathcal{N} \quad \text{on } \mathcal{T}_R.\end{aligned} \tag{76}$$

The same is true in the case that $\hat{U}(g) = \hat{C}\hat{V}(g)$ is antiunitary, with $\hat{V}(g)$ unitary. Then

$$\mathcal{N}\big(\Pi_{\mathcal{T}_R}(g)(\hat{A})\big) = \int \overline{V_K(g|x,x')A_K(x',y')}V_K(g|y',y)\,\mathrm{d}x'\,\mathrm{d}y' \tag{77}$$

and because

$$(\Pi_{\mathcal{H}}(g)\varphi)(x) = \int \overline{V_K(g|x,x')\varphi(x')}\,\mathrm{d}x' \tag{78}$$

and

$$\left(\Pi^C_{\mathcal{H}}(g)\varphi\right)(y) = \int V_K(g|y',y)\overline{\varphi(y')}\,\mathrm{d}y' \tag{79}$$

then (76) again holds.

Now let Z denote the unitary mapping from $L_2(\mathbb{C},\mathrm{d}x) \otimes L_2(\mathbb{C},\mathrm{d}y)$ to $\mathcal{K}_R$ defined by

$$Z = \mathcal{W}\mathcal{N}^{\dagger}. \tag{80}$$

It is not hard to see from (72) and (12) that

$$(Zf)(q,p) = \int f(q - x/2, q + x/2)\,\mathrm{e}^{\mathrm{i}px}\,\mathrm{d}x \tag{81}$$

with inverse acting as

$$(Z^{\dagger}F)(x,y) = \frac{1}{2\pi}\int F((x+y)/2, p)\,\mathrm{e}^{\mathrm{i}p(x-y)}\,\mathrm{d}p. \tag{82}$$

From (64) and (72), we have the isomorphism between $\Pi_{\mathcal{K}_R}$ and $\Pi_{\mathcal{H}} \otimes \Pi^C_{\mathcal{H}}$ in the form

$$\Pi_{\mathcal{K}_R} Z = Z\big(\Pi_{L_2(\mathbb{C},\mathrm{d}x)} \otimes \Pi^C_{L_2(\mathbb{C},\mathrm{d}y)}\big). \tag{83}$$

We say that $\Pi_{\mathcal{K}_R}$ is the Weyl–Wigner product of $\Pi_{\mathcal{H}}$ and $\Pi^C_{\mathcal{H}}$, denoted by

$$\Pi_{\mathcal{H}} \overset{W}{\otimes} \Pi^C_{\mathcal{H}}. \tag{84}$$

The reduction to irreducibles of the Weyl–Wigner product will evidently lead to the same Clebsch–Gordan series as the reduction of the usual tensor product $\Pi_{\mathcal{H}} \otimes \Pi^C_{\mathcal{H}}$, and the basis vectors on which the reduction is accomplished will be related by the intertwiner Z. We shall consider this further only in the context of example 4 (case A) in the next section, where the reductions can easily be worked out and compared.

6. Factorizing phase-space representations

Not every real, unitary representation $\Pi_{\mathcal{K}_R}$ of a group on the function space $\mathcal{K}_R$ is in the form of a Weyl–Wigner product. Only those representations forming subgroups of $\mathrm{Aut}(\mathbf{V}) < O(\mathcal{K}_R)$ have this form. In view of (60), the extra condition to be satisfied is

$$\Pi_{\mathcal{K}_R}(g)(A \star B) = \Pi_{\mathcal{K}_R}(g)(A) \star \Pi_{\mathcal{K}_R}(g)(B) \tag{85}$$

for all $A, B \in \mathcal{K}_R$ and all g in the group. Given a representation $\Pi_{\mathcal{K}_R}$ which is in $\mathrm{Aut}(\mathbf{V})$, and so does satisfy (85), it follows from Wigner's theorem that it must be possible to factorize $\Pi_{\mathcal{K}_R}$ as the Weyl–Wigner product of a representation $\Pi_{\mathcal{H}}$ on Hilbert space with its contragredient $\Pi^C_{\mathcal{H}}$, and that this representation $\Pi_{\mathcal{H}}$ will be in general a unitary or antiunitary ray representation of the underlying group.

We now examine how this factorization process can be put into effect, and begin by specializing to the case of a connected Lie group G with a unitary ray representation $\Pi_{\mathcal{H}}$ on $\mathcal{H}$, a corresponding real unitary representation $\Pi_{\mathcal{T}_R}$ on $\mathcal{T}_R$, and a corresponding real unitary representation $\Pi_{\mathcal{K}_R}$ on $\mathcal{K}_R$, with the isomorphisms (71).

Let $\hat{A}$ denote the self-adjoint linear operator acting on $\mathcal{H}$ which generates the one-parameter sub-representation of $\Pi_{\mathcal{H}}$ corresponding to a 1-parameter subgroup $H < G$. Let α denote the self-adjoint linear operator acting on $\mathcal{K}_R$ which generates the corresponding 1-parameter sub-representation of $\Pi_{\mathcal{K}_R}$. If we are given $\Pi_{\mathcal{H}}$, it is clear from (65) and (68) that we can determine $\Pi_{\mathcal{K}_R}$, and so, given $\hat{A}$, we can determine α in principle. We call this the 'direct problem'. More interesting, and less obvious, is that given $\Pi_{\mathcal{K}_R}$ and hence, implicitly, given α, we can solve the 'inverse problem' and determine $\hat{A}$. In this way we attempt to determine, one 1-parameter subgroup at a time, the ray representation $\Pi_{\mathcal{H}}$ from the real unitary representation $\Pi_{\mathcal{K}_R}$, in effect performing the factorization (71):

$$\Pi_{\mathcal{K}_R} = \Pi_{\mathcal{H}} \overset{W}{\otimes} \Pi_{\mathcal{H}}^{C}. \tag{86}$$

We consider two cases. In the first case, $(\hat{A} - a\hat{I}) \in \mathcal{T}_R$ for some real constant a; in the second case, $\hat{A} \in \mathcal{Q}_{WH}$.

Suppose first that we are given $(\hat{A} - a\hat{I}) \in \mathcal{T}_R$ for some real a, and hence a corresponding function A such that $(A - a) = \mathcal{W}(\hat{A} - a\hat{I}) \in \mathcal{K}_R$. Let $\tilde{A} = A - a$. We look for α in the form of an integral operator [44] on $\mathcal{K}_R$:

$$(\alpha B)(q_1, p_1) = \int \alpha_K(q_1, p_1, q_2, p_2) B(q_2, p_2)\, \mathrm{d}\Gamma_2. \tag{87}$$

The local (Lie algebraic) condition corresponding to the global (group-theoretic) condition (65) is

$$\alpha B = A \star B - B \star A = \tilde{A} \star B - B \star \tilde{A} = \mathrm{i}\{\tilde{A}, B\}_{GM} \tag{88}$$

from which it is easily checked that, as a consequence of (88),

$$\alpha(B \star C) = (\alpha B) \star C + B \star (\alpha C). \tag{89}$$

This is the local condition corresponding to (85). When (88) holds, we have from (21), for suitably smooth B, say $B \in \mathcal{J}_R$,

$$\begin{aligned} \int \alpha_K(q_1, p_1, q_2, p_2) B(q_2, p_2)\, \mathrm{d}\Gamma_2 = \frac{2\mathrm{i}}{\pi^2} \int [\sin\{2[p_1(q_2 - q_3) + p_2(q_3 - q_1) \\ + p_3(q_1 - q_2)]\}\tilde{A}(q_3, p_3)\, \mathrm{d}\Gamma_3] B(q_2, p_2)\, \mathrm{d}\Gamma_2 \end{aligned} \tag{90}$$

and so

$$\alpha_K(q_1, p_1, q_2, p_2) = \frac{2\mathrm{i}}{\pi^2} \int \sin\{2[p_1(q_2 - q_3) + p_2(q_3 - q_1) + p_3(q_1 - q_2)]\}\tilde{A}(q_3, p_3)\, \mathrm{d}\Gamma_3. \tag{91}$$

Then (87) and (91) define the action of α in terms of $\tilde{A}$ (and hence in terms of A or $\hat{A}$), thus solving the direct problem. Note that because $\tilde{A}$ is real, (91) implies

$$\overline{\alpha_K(q_1, p_1, q_2, p_2)} = \alpha_K(q_2, p_2, q_1, p_1) = -\alpha_K(q_1, p_1, q_2, p_2) \tag{92}$$

as required by selfadjointness of α, and the reality of $\Pi_{\mathcal{K}_R}$.

To solve the inverse problem, we must invert (91). This will only be possible if α_K is further constrained, because the conditions (92) only guarantee that α generates an element of $O(\mathcal{K}_R)$, and we require that α generates an element of $\mathrm{Aut}(\mathbf{V}) < O(\mathcal{K}_R)$. The further constraint is (89), but we wish to express it as a condition on α alone. We change variables in (91) and write it in the form

$$\begin{aligned}
&\alpha_K((u'-u)/2, (v'+v)/2, (u'+u)/2, (v'-v)/2)\\
&\qquad = -\frac{2\mathrm{i}}{\pi^2}\int \sin\{2v(q_3-u'/2)+2u(p_3-v'/2)\}\tilde{A}(q_3,p_3)\,\mathrm{d}\Gamma_3\\
&\qquad = -\frac{2\mathrm{i}}{\pi^2}\int \sin\{2vq_3+2up_3\}\tilde{A}(q_3+u'/2, p_3+v'/2)\,\mathrm{d}\Gamma_3\\
&\qquad = -\frac{\mathrm{i}}{2\pi^2}\int \sin\{vx+uy\}\tilde{A}((u'+x)/2, (v'+y)/2)\,\mathrm{d}x\,\mathrm{d}y
\end{aligned}\tag{93}$$

where

$$u = q_2 - q_1 \qquad v = p_1 - p_2 \qquad u' = q_2 + q_1 \qquad v' = p_1 + p_2. \tag{94}$$

Now (93) takes the form

$$R(u,v,u',v') = \int \sin(vx+uy)S(x,y,u',v')\,\mathrm{d}x\,\mathrm{d}y \tag{95}$$

where

$$\begin{aligned}
R(u,v,u',v') &= \alpha_K((u'-u)/2, (v'+v)/2, (u'+u)/2, (v'-v)/2)\\
S(x,y,u',v') &= -\frac{\mathrm{i}}{2\pi^2}\tilde{A}((u'+x)/2, (v'+y)/2).
\end{aligned}\tag{96}$$

Set

$$S^{(\pm)}(x,y,u',v') = \tfrac{1}{2}(S(x,y,u',v') \pm S(-x,-y,u',v')) \tag{97}$$

and note that (95) can be written as

$$\begin{aligned}
R(u,v,u',v') &= \int \sin(vx+uy)S^{(-)}(x,y,u',v')\,\mathrm{d}x\,\mathrm{d}y\\
&= -\mathrm{i}\int \mathrm{e}^{\mathrm{i}(vx+uy)}S^{(-)}(x,y,u',v')\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}\tag{98}$$

Inverting the double Fourier transform, we have

$$\begin{aligned}
S^{(-)}(x,y,u',v') &= \frac{\mathrm{i}}{(2\pi)^2}\int \mathrm{e}^{-\mathrm{i}(vx+uy)}R(u,v,u',v')\,\mathrm{d}u\,\mathrm{d}v\\
&= \frac{1}{(2\pi)^2}\int \sin(vx+uy)R(u,v,u',v')\,\mathrm{d}u\,\mathrm{d}v
\end{aligned}\tag{99}$$

using $R(-u,-v,u',v') = -R(u,v,u',v')$, which follows from (92). Reintroducing A from (96) and (97), we have

$$\begin{aligned}
&A((u'+x)/2, (v'+y)/2) - A((u'-x)/2, (v'-y)/2)\\
&\qquad = 2\mathrm{i}\int \sin(vx+uy)R(u,v,u',v')\,\mathrm{d}u\,\mathrm{d}v
\end{aligned}\tag{100}$$

and so

$$A(x,y) - A(0,0) = 2\mathrm{i}\int \sin(vx+uy)R(u,v,x,y)\,\mathrm{d}u\,\mathrm{d}v. \tag{101}$$

Then

$$\begin{aligned}
A((u'+x)/2, (v'+y)/2) - A(0,0) &= 2\mathrm{i}\int \sin(v(u'+x)/2\\
&\quad + u(v'+y)/2)R(u,v,(u'+x)/2,(v'+y)/2)\,\mathrm{d}u\,\mathrm{d}v
\end{aligned}\tag{102}$$

with a similar expression for $A((u'-x)/2,(v'-y)/2)-A(0,0)$. Subtracting this second expression from the first, and equating to the RHS of (100), we get

$$\int \sin(vx+uy)R(u,v,u',v')\,\mathrm{d}u\,\mathrm{d}v$$
$$= \int \sin(v(u'+x)/2+u(v'+y)/2)R(u,v,(u'+x)/2,(v'+y)/2)\,\mathrm{d}u\,\mathrm{d}v$$
$$- \int \sin(v(u'-x)/2+u(v'-y)/2)R(u,v,(u'-x)/2,(v'-y)/2)\,\mathrm{d}u\,\mathrm{d}v. \tag{103}$$

It is this condition, with R as in (96), that α_K must satisfy in addition to (92), if α is to generate an element of $\mathrm{Aut}(\mathbf{V}) < O(\mathcal{K}_R)$. To see this, and to solve the inverse problem, suppose now that we are given α_K satsifying (92) and (103), with R as in (96), and u, v, u', v' as in (94).

Set

$$A(x,y) = a + 2\mathrm{i}\int \sin(vx+uy)R(u,v,x,y)\,\mathrm{d}u\,\mathrm{d}v \tag{104}$$

where a is an arbitrary real constant, and check that A is real, and that (100) is satisfied. Then retrace the steps to recover (91), showing that A as given by (104) generates the automorphism associated with α. Note that A is only defined by α up to the arbitrary real constant a, so the corresponding unitary operator in $\Pi_{\mathcal{H}}$ is only defined up to a constant phase, as expected.

The treatment of this first case, with $\hat{A} \in \mathcal{T}_R$ and $A \in \mathcal{K}_R$, might be extended to the case of a general selfadjoint $\hat{A}$ and corresponding α, with a suitable extension of the interpretation of the integral formulae above to accommodate distributions. We only consider further the second case mentioned above, when $\hat{A} \in \mathcal{Q}_{WH}$. This can be treated more directly.

Suppose then that $\hat{A}$ is a Hermitian polynomial in the canonical operators $\hat{q}$, $\hat{p}$ and $\hat{I}$, as introduced in section 2. The corresponding A is a real polynomial in q, p and 1 of the same degree, and according to (88) and (26), α is a polynomial in q, p, $\partial/\partial q$ and $\partial/\partial p$ acting on suitably smooth $B \in \mathcal{K}_R$ (say $B \in \mathcal{J}_R$). This last polynomial is also of the same degree, except that it has no constant term. For example, corresponding to $A = q + a$, we have

$$\alpha B = \mathrm{i}\{q+a, B\}_{GM} = \mathrm{i}\frac{\partial B}{\partial p} \tag{105}$$

using (24), so that $A = q + a$ corresponds to $\alpha = \mathrm{i}\partial/\partial p$ for all values of the constant a.

When restricted to act on an invariant subspace of $\mathcal{H}$, the selfadjoint operators in Q_{WH},

$$\hat{X}_1 = \hat{I}, \quad \hat{X}_2 = \hat{q}, \quad \hat{X}_3 = \hat{p}, \quad \hat{X}_4 = \hat{q}^2, \quad \hat{X}_5 = \tfrac{1}{2}(\hat{q}\hat{p}+\hat{p}\hat{q}), \ldots \tag{106}$$

span an infinite-dimensional real Lie algebra $\mathcal{L}$. Choosing the coordinate representation $\mathcal{H} \cong L_2(\mathbb{C}, dx)$ as in section 2, we have the representation ξ of $\mathcal{L}$ on $\mathcal{G} < \mathcal{H}$ with

$$\xi(\hat{X}_1) = 1, \quad \xi(\hat{X}_2) = x, \quad \xi(\hat{X}_3) = -\mathrm{i}\frac{\partial}{\partial x}, \quad \xi(\hat{X}_4) = x^2, \quad \xi(\hat{X}_5) = -\mathrm{i}\left(x\frac{\partial}{\partial x} + \frac{1}{2}\right), \ldots. \tag{107}$$

The mapping Ξ, carrying selfadjoint operators $\hat{A}$ in Q_{WH} into corresponding selfadjoint operators α acting on $\mathcal{K}_R$, defines a representation of $\mathcal{L}$ on $\mathcal{J}_R$. This can be seen explicitly from (88), which gives for any $B \in \mathcal{J}_R$,

$$\Xi(\hat{X}_i)\Xi(\hat{X}_j)B = -\{\mathcal{W}(\hat{X}_i), \{\mathcal{W}(\hat{X}_j), B\}_{GM}\}_{GM} \tag{108}$$

so that

$$[\Xi(\hat{X}_i), \Xi(\hat{X}_j)]B = \{\{\mathcal{W}(\hat{X}_i), \mathcal{W}(\hat{X}_j)\}_{GM}, B\}_{GM}$$
$$= \{\mathcal{W}([\hat{X}_i, \hat{X}_j]), B\}_{GM}$$
$$= \Xi([\hat{X}_i, \hat{X}_j])B \tag{109}$$

Table 1. Corrresponding $\hat{A}$, A and α. Subscripts on P indicate partial derivatives, and $V^{(n)}$ denotes the nth derivative of V.

$\hat{A}$	A	α
$\mathcal{W}^{-1}(P(q,p))$	$P(q,p)$	$\mathrm{i}\left(P_q \frac{\partial}{\partial p} - P_p \frac{\partial}{\partial q}\right) - \frac{\mathrm{i}}{3!4}\left(P_{qqq}\frac{\partial^3}{\partial p^3} - 3P_{qqp}\frac{\partial^3}{\partial q \partial p^2} + 3P_{qpp}\frac{\partial^3}{\partial q^2 \partial p} - P_{ppp}\frac{\partial^3}{\partial q^3}\right) + \frac{\mathrm{i}}{5!4^2}\left(P_{qqqqq}\frac{\partial^5}{\partial p^5} - \ldots\right.$
$\hat{I}$	1	0
$\hat{q}$	q	$\mathrm{i}\frac{\partial}{\partial p}$
$\hat{p}$	p	$-\mathrm{i}\frac{\partial}{\partial q}$
$\hat{q}^2$	q^2	$2\mathrm{i}q\frac{\partial}{\partial p}$
$\hat{p}^2$	p^2	$-2\mathrm{i}p\frac{\partial}{\partial q}$
$\frac{1}{2}(\hat{q}\hat{p}+\hat{p}\hat{q})$	qp	$\mathrm{i}p\frac{\partial}{\partial p} - \mathrm{i}q\frac{\partial}{\partial q}$
$\hat{q}^3$	q^3	$3\mathrm{i}q^2\frac{\partial}{\partial p} - \frac{1}{4}\mathrm{i}\frac{\partial^3}{\partial p^3}$
$\hat{p}^3$	p^3	$-3\mathrm{i}p^2\frac{\partial}{\partial q} + \frac{1}{4}\mathrm{i}\frac{\partial^3}{\partial q^3}$
$\hat{q}\hat{p}\hat{q}$	q^2p	$2\mathrm{i}qp\frac{\partial}{\partial p} - \mathrm{i}q^2\frac{\partial}{\partial q} + \frac{1}{8}\mathrm{i}\frac{\partial^3}{\partial q\partial p^2}$
$\hat{p}\hat{q}\hat{p}$	qp^2	$\mathrm{i}p^2\frac{\partial}{\partial p} - 2\mathrm{i}qp\frac{\partial}{\partial q} - \frac{1}{8}\mathrm{i}\frac{\partial^3}{\partial q^2\partial p}$
$V(\hat{q})$	$V(q)$	$\mathrm{i}V^{(1)}(q)\frac{\partial}{\partial p} - \frac{\mathrm{i}}{3!4}V^{(3)}(q)\frac{\partial^3}{\partial p^3} + \frac{\mathrm{i}}{5!4^2}V^{(5)}(q)\frac{\partial^5}{\partial p^5} - \cdots$

using the antisymmetry property of the Groenewold–Moyal bracket, and the associated Jacobi identity. Thus, when all the generators of the representation $\Pi_{\mathcal{K}_R}$ of the group G belong to Q_{WH}, they provide a representation on $\mathcal{J}_R$ of the Lie algebra of G. Using (88), we find explicitly that

$$\Xi(\hat{X}_1) = 0, \quad \Xi(\hat{X}_2) = \mathrm{i}\frac{\partial}{\partial p}, \quad \Xi(\hat{X}_3) = -\mathrm{i}\frac{\partial}{\partial q}, \quad \Xi(\hat{X}_4) = 2\mathrm{i}q\frac{\partial}{\partial p}, \ldots. \tag{110}$$

In table 1 we list some corresponding $\hat{A}$, $A = \mathcal{W}(\hat{A})$ and $\alpha = \Xi(\hat{A})$ obtained using (88). Note that every α is formally Hermitian and pure imaginary, as required by the unitarity and reality of Π. The extension of the operators $\hat{A}$ and α from Hermitian polynomials on $\mathcal{G}$ and $\mathcal{J}_R$ respectively, to selfadjoint operators on appropriate domains in $\mathcal{H}$ and $\mathcal{K}_R$ is straightforward.

Note that Ξ defines a Lie algebra homomorphism but not an algebra homomorphism: it does not define a representation of the enveloping algebra of the Heisenberg–Weyl Lie algebra. For example, as can be seen from the table, $\Xi(\hat{X}_2)\Xi(\hat{X}_3) + \Xi(\hat{X}_3)\Xi(\hat{X}_2) \neq \Xi(\hat{X}_2\hat{X}_3 + \hat{X}_3\hat{X}_2)$.

Corresponding to (111), the representation Ξ of $\mathcal{L}$ is isomorphic to the tensor product of the representation ξ on $L_2(\mathbb{C}, \mathrm{d}x)$ as in (107) and its contragredient ξ^C on $L_2(\mathbb{C}, \mathrm{d}y)$, so that on $\mathcal{J}_R$,

$$\Xi = Z(\xi \otimes \xi^c)Z^\dagger \tag{111}$$

where Z is the unitary transformation (81), and

$$\begin{aligned} &\xi^C(\hat{X}_1) = -1, \quad \xi^C(\hat{X}_2) = -y, \quad \xi^C(\hat{X}_3) = -\mathrm{i}\frac{\partial}{\partial y}, \\ &\xi^C(\hat{X}_4) = -y^2, \quad \xi^C(\hat{X}_5) = -\mathrm{i}\left(y\frac{\partial}{\partial y} + \frac{1}{2}\right), \ldots. \end{aligned} \tag{112}$$

The rule in going from (107) to (112) is that each real expression attracts a minus sign, whereas each pure imaginary expression does not. Straightforward calculation shows that

$$\begin{aligned} ZxZ^\dagger &= q + \frac{1}{2}\mathrm{i}\frac{\partial}{\partial p} & ZyZ^\dagger &= q - \frac{1}{2}\mathrm{i}\frac{\partial}{\partial p} \\ Z\mathrm{i}\frac{\partial}{\partial x}Z^\dagger &= -p + \frac{1}{2}\mathrm{i}\frac{\partial}{\partial q} & Z\mathrm{i}\frac{\partial}{\partial y}Z^\dagger &= p + \frac{1}{2}\mathrm{i}\frac{\partial}{\partial q} \end{aligned} \tag{113}$$

with inverses

$$\begin{aligned} Z^\dagger qZ &= \frac{1}{2}(x+y) & Z^\dagger\mathrm{i}\frac{\partial}{\partial p}Z &= x - y \\ Z^\dagger pZ &= \frac{1}{2}\left(-\mathrm{i}\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right) & Z^\dagger\mathrm{i}\frac{\partial}{\partial q}Z &= \mathrm{i}\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y} \end{aligned} \tag{114}$$

from which one can easily deduce that, corresponding to the monomial $A = q^m p^n$ and its image $\hat{A}$ under $\mathcal{W}^{-1}$ as in (23), we have

$$\begin{aligned} \Xi(\mathcal{W}^{-1}(q^m p^n)) = \frac{1}{2^m}\sum_{r=0}^{m} C_r^m \Bigg[&\left(q + \frac{1}{2}\mathrm{i}\frac{\partial}{\partial p}\right)^{m-r} \left(p - \frac{1}{2}\mathrm{i}\frac{\partial}{\partial q}\right)^{n} \left(q + \frac{1}{2}\mathrm{i}\frac{\partial}{\partial p}\right)^{r} \\ &- \left(q - \frac{1}{2}\mathrm{i}\frac{\partial}{\partial p}\right)^{m-r} \left(p + \frac{1}{2}\mathrm{i}\frac{\partial}{\partial q}\right)^{n} \left(q - \frac{1}{2}\mathrm{i}\frac{\partial}{\partial p}\right)^{r} \Bigg]. \end{aligned} \tag{115}$$

Related formulae have been presented recently by Hakioglu and Dragt [45].

The extra constraint corresponding to (103), which α must satisfy in order to generate an element of Aut(**V**), is simply this: only those polynomials in $q, p, \partial/\partial q$ and $\partial/\partial p$ which are real linear combinations of those in (115), represent possible α. For example, $\alpha = \mathrm{i}q^2\partial/\partial p$ is not allowed; it generates an element of $O(\mathcal{K}_R)$ that is not in Aut(**V**). A more straightforward test of a candidate α is to evaluate $(Z^\dagger \alpha Z)$ using (114). The resulting operator on $L_2(\mathbb{C}, \mathrm{d}x) \otimes L_2(\mathbb{C}, \mathrm{d}y)$ must have the form $\hat{A}(x, -\mathrm{i}\partial/\partial x) - \hat{A}(y, -\mathrm{i}\partial/\partial y)$ for some Hermitian polynomial operator $\hat{A}(x, -\mathrm{i}\partial/\partial x)$ on $L_2(\mathbb{C}, \mathrm{d}x)$.

The direct and inverse problems in this case are solved simply by consulting table 1, extended if necessary to higher degrees using (88) or (115). That is to say, given $\hat{A}$ (or A), read off α; given α, read off $\hat{A}$ (or A) up to the addition of an arbitrary real constant multiple of $\hat{I}$ (or 1). Alternatively, to solve the inverse problem, proceed as in the preceding paragraph to identify $\hat{A}$ (to within a constant multiple of the identity operator).

7. Examples

It is informative in the first two examples to use dimensional variables, introducing factors of $\hbar$ in the appropriate places.

7.1. The Heisenberg–Weyl group

Elements of this two-parameter, real, Abelian Lie group are labelled $g(a_1, a_2)$, where a_1 and a_2 take all real values, and the product rule is

$$g(a_1, a_2)g(b_1, b_2) = g(a_1 + b_1, a_2 + b_2). \tag{116}$$

The real, true, unitary representation on $\mathcal{K}_R$ in this case has the form

$$\big(\Pi_{\mathcal{K}_R}(a_1, a_2)F\big)(q, p) = F(q + a_1, p - a_2) \tag{117}$$

with the associated generators

$$\alpha_1 = -\mathrm{i}\frac{\partial}{\partial q} \qquad \alpha_2 = \mathrm{i}\frac{\partial}{\partial p} \tag{118}$$

satisfying on $\mathcal{J}_R$ the commutation relation

$$[\alpha_1, \alpha_2] = 0. \tag{119}$$

Note that Planck's constant does not appear in $\Pi_{\mathcal{K}_R}$.

Turning to the 'factorization' (86), we have from (117) and (82) that

$$\begin{aligned}\left(Z^\dagger \Pi_{\mathcal{K}_R}(a_1, a_2)F\right)(x, y) &= \frac{1}{2\pi\hbar}\int F\left(\frac{x+y}{2} + a_1, p - a_2\right) \mathrm{e}^{\mathrm{i}p(x-y)/\hbar}\,\mathrm{d}p \\ &= \frac{1}{2\pi\hbar}\mathrm{e}^{\mathrm{i}a_2(x-y)/\hbar}\int F\left(\frac{x+y}{2} + a_1, p\right) \mathrm{e}^{\mathrm{i}p(x-y)/\hbar}\,\mathrm{d}p.\end{aligned} \tag{120}$$

Setting

$$(Z^\dagger F)(x, y) = f(x, y) \tag{121}$$

so that, from (82),

$$f(x, y) = \frac{1}{2\pi\hbar}\int F\left(\frac{x+y}{2}, p\right) \mathrm{e}^{\mathrm{i}p(x-y)/\hbar}\,\mathrm{d}p \tag{122}$$

we have from (120) that

$$\left(\left(\Pi_{L_2(\mathbb{C},\mathrm{d}x)} \otimes \Pi^C_{L_2(\mathbb{C},\mathrm{d}y)}\right)f\right)(x, y) = \mathrm{e}^{\mathrm{i}a_2(x-y)/\hbar} f(x + a_1, y + a_1). \tag{123}$$

From this we see that a possible factorization is obtained by taking the action of $\Pi_{L_2(\mathbb{C},\mathrm{d}x)}$ and $\Pi^C_{L_2(\mathbb{C},\mathrm{d}y)}$ on $u \in L_2(\mathbb{C}, \mathrm{d}x)$ and $v \in L_2(\mathbb{C}, \mathrm{d}y)$, respectively, to be

$$\begin{aligned}\left(\Pi_{L_2(\mathbb{C},\mathrm{d}x)}(a_1, a_2)u\right)(x) &= \mathrm{e}^{\mathrm{i}\omega(a_1,a_2)}\,\mathrm{e}^{\mathrm{i}a_2x/\hbar}u(x + a_1) \\ \left(\Pi^C_{L_2(\mathbb{C},\mathrm{d}y)}(a_1, a_2)v\right)(y) &= \mathrm{e}^{-\mathrm{i}\omega(a_1,a_2)}\,\mathrm{e}^{-\mathrm{i}a_2y/\hbar}v(y + a_1)\end{aligned} \tag{124}$$

where ω is real-valued. It is then readily checked that $\Pi_{L_2(\mathbb{C},\mathrm{d}x)}$ is a projective representation of the Abelian group (116), whatever the form of ω. Different choices for ω correspond to different choices, from the same cohomology class, of the cocycle associated with projective representations of the group, and do not differ in a significant way. We may say that, up to the phase ω, we have recovered in (124) the usual projective unitary representation on $\mathcal{H}$, realized as $L_2(\mathbb{C}, \mathrm{d}x)$.

We can also consider this example from the Lie algebraic viewpoint. Using (114), now with appropriate factors of $\hbar$ inserted, we have at once from (118) that

$$Z^\dagger\alpha_1 Z = -\mathrm{i}\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y} \qquad Z^\dagger\alpha_2 Z = \frac{x}{\hbar} - \frac{y}{\hbar} \tag{125}$$

from which we have

$$\hat{A}_1 = \hat{p} = -\mathrm{i}\hbar\frac{\partial}{\partial x} + p_0 \qquad \hat{A}_2 = \hat{q} = x + q_0 \tag{126}$$

where q_0 and p_0 are arbitrary constants. Then

$$[\hat{q}, \hat{p}] = \mathrm{i}\hbar \tag{127}$$

on $\mathcal{G}$, and $\hat{A}_1, \hat{A}_2$ are equivalent to the usual canonical operators there. Note that $\hbar$ appears on the RHS of (127) as the parameter associated with a central extension of the Lie algebra in going from (119) to (127). There is no $\hbar$ in $\Pi_{\mathcal{K}_R}$, but there is in $\Pi_{\mathcal{H}}$. Evidently its appearance comes from the unitary transformation Z, or equivalently, from the Weyl–Wigner transform. Note also that from this point of view, $\hbar$ is an arbitrary parameter; the factorization (86) of

$\Pi_{\mathcal{K}_R}$ works for any value of $\hbar$ in Z in this case, and the one to be chosen ultimately is a matter for physics to decide.

The Heisenberg–Weyl algebra can be generalized in an obvious way, showing that Lie algebras with polynomial elements of arbitrarily high degree can arise in the phase-space formalism. Consider the $(N+1)$-dimensional real Lie algebra with selfadjoint representation generated by

$$\alpha_1 = -\mathrm{i}\frac{\partial}{\partial q} \tag{128}$$

together with

$$\beta_1 = \mathrm{i}\frac{\partial}{\partial p}, \quad \beta_2 = \mathrm{i}q\frac{\partial}{\partial p}, \quad \beta_3 = \frac{1}{2}\mathrm{i}q^2\frac{\partial}{\partial p} - \frac{1}{24}\mathrm{i}\frac{\partial^3}{\partial p^3}, \ldots, \beta_N \qquad \text{that is}$$

$$\beta_n = \frac{2}{n!}\sum_{m=1}^{n}\left(\frac{\mathrm{i}}{2}\right)^{n-m} C_m^n q^m \frac{\partial^{n-m}}{\partial p^{n-m}} \quad n = 1, 2, \ldots, N \tag{129}$$

where C_m^n is the binomial coefficient as in (22), and the sum is restricted to odd values of $n-m$. In generalization of (126), we find that the corresponding operators on $\mathcal{H}$ are

$$\hat{A}_1 = -\mathrm{i}\hbar\frac{\partial}{\partial x} \qquad \hat{B}_n = \frac{x^n}{n!} \qquad n = 1, 2, \ldots, N \tag{130}$$

up to the addition of arbitrary real constants. Once again, the Lie algebra generated by $\hat{A}_1$ and the $\hat{B}_n$ is a central extension of the Lie algebra generated by α_1 and the β_n, and Planck's constant appears as the extension parameter.

7.2. *The Galilei group*

For a system with one degree of freedom, this group is a three-parameter real Lie group [55] with elements $g(a_1, a_2, a_3)$, where a_1, a_2 and a_3 take all real values, and the product rule is

$$g(a_1, a_2, a_3)g(b_1, b_2, b_3) = g(a_1 + b_1, a_2 + b_2, a_3 + b_3 + b_2 a_1). \tag{131}$$

Consider the true, real, unitary representation on $\mathcal{K}_R$ defined by

$$\left(\Pi_{\mathcal{K}_R}(a_1, a_2, a_3)F\right)(q, p) = F\left(q - \frac{a_1}{m}p - a_2 a_1 - a_3, p + m a_2\right) \tag{132}$$

with associated generators

$$\alpha_1 = -\mathrm{i}\frac{p}{m}\frac{\partial}{\partial q} \qquad \alpha_2 = \mathrm{i}m\frac{\partial}{\partial p} \qquad \alpha_3 = -\mathrm{i}\frac{\partial}{\partial q} \tag{133}$$

satisfying the commutation relations

$$[\alpha_1, \alpha_2] = -\mathrm{i}\alpha_3 \qquad [\alpha_2, \alpha_3] = 0 \qquad [\alpha_1, \alpha_3] = 0. \tag{134}$$

Note in this case that m appears as a parameter in the action (132) of the group representation, though not in the commutation relations (134).

It is easiest to perform a factorization in this case after realizing $\mathcal{H}$ and its dual as $L_2(\mathbb{C}, \mathrm{d}r)$ and $L_2(\mathbb{C}, \mathrm{d}s)$, respectively, where r and s are 'momentum' variables. In place of (81) and (82), we have

$$\begin{aligned} F(q, p) &= \int \tilde{f}\left(p - \frac{r}{2}, p + \frac{r}{2}\right) \mathrm{e}^{-\mathrm{i}rq/\hbar}\,\mathrm{d}r = (\tilde{Z}\tilde{f})(q, p) \\ \tilde{f}(r, s) &= \frac{1}{2\pi}\int F\left(q, \frac{r+s}{2}\right) \mathrm{e}^{-\mathrm{i}q(r-s)/\hbar}\,\mathrm{d}q = (\tilde{Z}^{\dagger}F)(r, s). \end{aligned} \tag{135}$$

Considering (132), we then have

$$\left(\tilde{Z}^{\dagger}\Pi_{\mathcal{K}_R}F\right)(r,s)=\frac{1}{2\pi}\int F\left(q-\frac{a_1}{m}\frac{r+s}{2}-a_2a_1-a_3,\ \frac{r+s}{2}+ma_2\right)\mathrm{e}^{-\mathrm{i}q(r-s)/\hbar}\,\mathrm{d}q \tag{136}$$

so that

$$\left(\left(\Pi_{L_2(\mathbb{C},\mathrm{d}r)}\otimes\Pi^{C}_{L_2(\mathbb{C},\mathrm{d}s)}\right)\tilde{f}\right)(r,s)=\left(\tilde{Z}^{\dagger}\Pi_{\mathcal{K}_R}F\right)(r,s)$$

$$=\frac{1}{2\pi}\int\tilde{f}\left(\frac{r+s}{2}+ma_2-\frac{r'}{2},\ \frac{r+s}{2}+ma_2+\frac{r'}{2}\right)$$

$$\mathrm{e}^{-\mathrm{i}r'(q+\frac{-a_1}{m}\frac{r+s}{2}-a_2a_1-a_3)/\hbar}\,\mathrm{e}^{-\mathrm{i}q(r-s)/\hbar}\,\mathrm{d}q\,\mathrm{d}r',$$

$$=\tilde{f}(r+ma_2,s+ma_2)\,\mathrm{e}^{-\mathrm{i}\frac{a_1}{m}(r^2-s^2)}\,\mathrm{e}^{-\mathrm{i}(a_2a_1+a_3)(r-s)}. \tag{137}$$

We see that a possible factorization has

$$\begin{aligned}\left(\Pi_{L_2(\mathbb{C},\mathrm{d}r)}(a_1,a_2,a_3)u\right)(r)&=\mathrm{e}^{\mathrm{i}\omega(a_1,a_2,a_3)}\,\mathrm{e}^{-\mathrm{i}\frac{a_1}{m}r^2}\,\mathrm{e}^{-\mathrm{i}(a_2a_1+a_3)r}u(r+ma_2)\\ \left(\Pi_{L_2(\mathbb{C},\mathrm{d}s)^C}(a_1,a_2,a_3)v\right)(s)&=\mathrm{e}^{-\mathrm{i}\omega(a_1,a_2,a_3)}\,\mathrm{e}^{\mathrm{i}\frac{a_1}{m}s^2}\,\mathrm{e}^{\mathrm{i}(a_2a_1+a_3)s}v(s+ma_2)\end{aligned} \tag{138}$$

which, up to the arbitrary phase ω, is the familiar action of the unitary ray representation of the Galilei group in the momentum space realization of Hilbert space, and of its contragredient representation.

From the Lie algebraic viewpoint, we find from (114)

$$\begin{aligned}Z^{\dagger}\alpha_1Z&=-\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}+\frac{\hbar}{2m}\frac{\partial^2}{\partial y^2}\qquad Z^{\dagger}\alpha_2Z=\frac{m}{\hbar}x-\frac{m}{\hbar}y\\ Z^{\dagger}\alpha_3Z&=-\mathrm{i}\frac{\partial}{\partial x}-\mathrm{i}\frac{\partial}{\partial y}\end{aligned} \tag{139}$$

from which we deduce that, in the coordinate reopresentation now,

$$\begin{aligned}\hat{A}_1&=\hat{H}=-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+e_0\qquad \hat{A}_2=\hat{K}=m(x+q_0)\\ \hat{A}_3&=\hat{p}=-\mathrm{i}\hbar\frac{\partial}{\partial x}+p_0\end{aligned} \tag{140}$$

where e_0, q_0 and p_0 are arbitrary constants. Then $\hat{H}$, $\hat{K}$ and $\hat{p}$ are equivalent to the usual Hamiltonian, boost and momentum operators for the free particle in one dimension, and satisfy on $\mathcal{G}$ the familiar relations

$$[\hat{H},\hat{K}]=-\mathrm{i}\hbar\hat{p}\qquad[\hat{H},\hat{p}]=0\qquad[\hat{K},\hat{p}]=\hbar m. \tag{141}$$

Comparing with (134), we see the appearance of m in (141), associated with a central extension of the Lie algebra of the Galilei group. Although $\Pi_{\mathcal{K}_R}$ is a true representation of the group, associated with the commutation relations (134) in which no m appears, nevertheless m is a parameter in $\Pi_{\mathcal{K}_R}$, enabling the factorization (138) to take place, and the m to appear in (141).

7.3. Two one-parameter groups

Consider the 1-parameter transformation group acting on Γ with generator α, whose kernel as in (87) is given by

$$\alpha_K(q_1,p_1,q_2,p_2)$$

$$=\mathrm{i}\sin[(1+\epsilon)(p_1q_2-p_2q_1)-(1-\epsilon)(q_1p_1-q_2p_2)]\,\mathrm{e}^{-(q_1-q_2)^2/\tau-(p_1-p_2)^2/\sigma}$$

$$\Rightarrow R(u,v,u',v')=\mathrm{i}\sin(uv'+\epsilon u'v)\,\mathrm{e}^{-u^2/\tau-v^2/\sigma} \tag{142}$$

where τ and σ are positive constants, and $\epsilon = \pm 1$. Then the constraints (92) are satisfied, so that the corresponding operator α generates an element of $O(\mathcal{K}_R)$. Now we have

$$2\mathrm{i}\int \sin(vx + uy)R(u, v, u', v')\,\mathrm{d}u\,\mathrm{d}v$$
$$= 2\pi\sqrt{\tau\sigma}\big(\mathrm{e}^{-((y+v')/2)^2\tau}\,\mathrm{e}^{-((x+\epsilon u')/2)^2\sigma} - \mathrm{e}^{-((y-v')/2)^2\tau}\,\mathrm{e}^{-((x-\epsilon u')/2)^2\sigma}\big) \qquad (143)$$

and (103) is seen to be satisfied if $\epsilon = 1$, but not satisfied if $\epsilon = -1$. If $\epsilon = 1$, (104) gives

$$A(q, p) = a + 2\pi\sqrt{\sigma\tau}\,\mathrm{e}^{-\sigma q^2 - \tau p^2} \qquad (144)$$

with a arbitrary. If $\epsilon = -1$, the element of $O(\mathcal{K}_R)$ generated by α is not an element of Aut($\mathbf{V}$), and no A exists.

7.4. The Lie algebra sp(2, R): case A

We consider the representation on $\mathcal{K}_R$ with

$$\alpha_1 = \frac{\mathrm{i}}{2}\left(p\frac{\partial}{\partial p} - q\frac{\partial}{\partial q}\right) \qquad \alpha_2 = \frac{\mathrm{i}}{2}\left(q\frac{\partial}{\partial p} + p\frac{\partial}{\partial q}\right)$$
$$\alpha_3 = \frac{\mathrm{i}}{2}\left(q\frac{\partial}{\partial p} - p\frac{\partial}{\partial q}\right) \qquad (145)$$

satisfying

$$[\alpha_1, \alpha_2] = -\mathrm{i}\alpha_3 \qquad [\alpha_2, \alpha_3] = \mathrm{i}\alpha_1 \qquad [\alpha_3, \alpha_1] = \mathrm{i}\alpha_2. \qquad (146)$$

Performing the factorization as in the previous examples, we get in this case (up to the addition of constant terms, which can be removed by redefinitions)

$$\hat{A}_1 = -\frac{\mathrm{i}}{2}\left(x\frac{\partial}{\partial x} + 1\right) \qquad \hat{A}_2 = \frac{1}{4}\left(x^2 + \frac{\partial^2}{\partial x^2}\right)$$
$$\hat{A}_3 = \frac{1}{4}\left(x^2 - \frac{\partial^2}{\partial x^2}\right) \qquad (147)$$

satisfying relations corresponding to (146). This is the representation associated with the simple harmonic oscillator. Each $\hat{A}_i$ is quadratic in the canonical operators on $\mathcal{G}$, and the quadratic Casimir operator for the Lie algebra has the value

$$-\hat{A}_1^2 - \hat{A}_2^2 + \hat{A}_3^2 = -\tfrac{3}{16}. \qquad (148)$$

No non-trivial central extensions are involved in this case, which has been described elsewhere in the phase-space context, from a slightly different point of view [45].

It is interesting in this example to compare the reductions to irreducible components of the Weyl–Wigner product $\Pi_{\mathcal{K}_R}$, and the usual tensor product $\Pi_{\mathcal{H}} \otimes \Pi_{\mathcal{H}}^C$, which can be found explicitly in both representations. In $\Pi_{\mathcal{K}_R}$ we look for the common eigenfunctions of

$$\Lambda^2 = -\alpha_1^2 - \alpha_2^2 + \alpha_3^2 + \frac{1}{4}$$
$$= \frac{1}{4}\left(p^2\frac{\partial^2}{\partial p^2} + 3p\frac{\partial}{\partial p} + q^2\frac{\partial^2}{\partial q^2} + 3q\frac{\partial}{\partial q} + 2pq\frac{\partial^2}{\partial q\partial p} + 1\right)$$
$$= \left(\frac{1}{2}\left(r\frac{\partial}{\partial r} + 1\right)\right)^2 \qquad (149)$$

and

$$\alpha_3 = \frac{\mathrm{i}}{2}\left(q\frac{\partial}{\partial p} - p\frac{\partial}{\partial q}\right) = \frac{1}{2}\mathrm{i}\frac{\partial}{\partial\theta} \qquad (150)$$

where in (150) and the last line of (149) we have introduced polar variables in the phase plane

$$q = r\cos(\theta) \qquad p = r\sin(\theta) \qquad 0 \leqslant r \leqslant \infty \quad 0 \leqslant \theta < 2\pi. \tag{151}$$

The common (generalized, unnormalized) eigenfunctions of Λ and α_3 are then seen to be

$$\Phi_{\lambda,m}(r,\theta) = \frac{\mathrm{e}^{-\mathrm{i}\lambda\ln r}}{r}\,\mathrm{e}^{-\mathrm{i}m\theta} \qquad -\infty < \lambda < \infty \quad m = 0, \pm 1, \pm 2, \ldots. \tag{152}$$

In fact, there are two irreducible representations here for each value of λ, one with all even integer values of m, and one with all odd integer values.

In $\Pi_{\mathcal{H}} \otimes \Pi^C_{\mathcal{H}}$, with $\hat{A}_i(x, -\mathrm{i}\partial/\partial x) = \hat{A}_i$ as in (147), we seek the common eigenfunctions $\Psi_{\lambda',m'}(x, y)$ of

$$\begin{aligned} -\big(\hat{A}_1(x, -\mathrm{i}\partial/\partial x) - \overline{\hat{A}_1(y, -\mathrm{i}\partial/\partial y)}\big)^2 - \big(\hat{A}_2(x, -\mathrm{i}\partial/\partial x) - \overline{\hat{A}_2(y, -\mathrm{i}\partial/\partial y)}\big)^2 \\ + \big(\hat{A}_3(x, -\mathrm{i}\partial/\partial x) - \overline{\hat{A}_3(y, -\mathrm{i}\partial/\partial y)}\big)^2 + \tfrac{1}{4} \\ = -\tfrac{1}{4}[\mathrm{i}(ab - a^\dagger b^\dagger)]^2 = -\Lambda'^2 \qquad \text{say} \end{aligned} \tag{153}$$

and

$$J_3 = \hat{A}_3(x) - \hat{A}_3(y) = \tfrac{1}{2}(a^\dagger a - b^\dagger b) \tag{154}$$

where we have introduced the boson operators

$$\begin{aligned} a &= \frac{1}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right) \qquad & b &= \frac{1}{\sqrt{2}}\left(y + \frac{\partial}{\partial y}\right) \\ a^\dagger &= \frac{1}{\sqrt{2}}\left(x - \frac{\partial}{\partial x}\right) \qquad & b^\dagger &= \frac{1}{\sqrt{2}}\left(y - \frac{\partial}{\partial y}\right). \end{aligned} \tag{155}$$

When m' is nonnegative, these eigenfunctions have the (unnormalized) form

$$\Psi_{\lambda',m'}(x, y) = a^{\dagger m'} W_{\frac{\lambda'}{2},\frac{m'}{2}}(2a^\dagger b^\dagger)\varphi_0(x, y) \tag{156}$$

where $\varphi_0(x, y) = \exp[-(x^2 + y^2)/2]$ is the 'vacuum vector,' annihilated by a and b, and $W_{\mu\nu}$ denotes a Whittaker function [50]. When m' is negative, the prefactor $a^{\dagger m'}$ on the RHS must be replaced by $b^{\dagger -m'}$. Again there are two irreducible representations here for each value of λ', one with all even integer values of m', and one with all odd integer values. The basis functions $\Phi_{\lambda,m}(q, p)$ and $\Psi_{\lambda,m}(x, y)$ must, of course, be related as in (81) and (82), but it is by no means obvious that this is so.

7.5. The Lie algebra $sp(2, R)$: case B

As another example where generators of higher degree than quadratic in the underlying variables occur, we consider the selfadjoint representation of $sp(2, R)$ on $\mathcal{K}_R$ with

$$\begin{aligned} \alpha_1 &= \frac{1}{2}\mathrm{i}\frac{\partial}{\partial p} - \frac{1}{2}\mathrm{i}p^2\frac{\partial}{\partial p} + \mathrm{i}\left(qp - \frac{1}{2}a\right)\frac{\partial}{\partial q} + \frac{1}{8}\mathrm{i}\frac{\partial^3}{\partial q^2\partial p} \\ \alpha_2 &= -\mathrm{i}\left(q\frac{\partial}{\partial q} - p\frac{\partial}{\partial p}\right) \\ \alpha_3 &= -\frac{1}{2}\mathrm{i}\frac{\partial}{\partial p} - \frac{1}{2}\mathrm{i}p^2\frac{\partial}{\partial p} + \mathrm{i}\left(qp - \frac{1}{2}a\right)\frac{\partial}{\partial q} + \frac{1}{8}\mathrm{i}\frac{\partial^3}{\partial q^2\partial p} \end{aligned} \tag{157}$$

again satisfying (146), with a an arbitrary real parameter. Performing the factorization, we get in this case (again after redefinitions, where necessary)

$$\begin{aligned} \hat{A}_1 &= \frac{1}{2}\left(x + x\frac{\partial^2}{\partial x^2} + (1 - \mathrm{i}a)\frac{\partial}{\partial x}\right) \qquad \hat{A}_2 = -\mathrm{i}\left(x\frac{\partial}{\partial x} + \frac{1}{2}(1 - \mathrm{i}a)\right) \\ \hat{A}_3 &= -\frac{1}{2}\left(x - x\frac{\partial^2}{\partial x^2} - (1 - \mathrm{i}a)\frac{\partial}{\partial x}\right) \end{aligned} \tag{158}$$

satisfying relations corresponding to (146). The quadratic Casimir in this case has the value

$$-\hat{A}_1^2 - \hat{A}_2^2 + \hat{A}_3^2 = -\tfrac{1}{4}(a^2+1) \tag{159}$$

showing that, whatever the real value of a, this selfadjoint representation on $\mathcal{H}$ is inequivalent to the one associated with (147) and (148). Again, no non-trivial central extensions are involved in this case.

7.6. Time reversal

The group has two elements g and e (identity) with $g^2 = e$. The real, true, unitary representation $\Pi_{\mathcal{K}_R}$ acts as

$$\left(\Pi_{\mathcal{K}_R}(g)F\right)(q,p) = F(q,-p) \qquad \left(\Pi_{\mathcal{K}_R}(e)F\right)(q,p) = F(q,p) \tag{160}$$

for every $F \in \mathcal{K}_R$. We have from (82),

$$\begin{aligned}\left(Z^\dagger\Pi_{\mathcal{K}_R}(g)F\right)(x,y) &= \frac{1}{2\pi}\int F\left(\frac{x+y}{2},-p\right)\mathrm{e}^{\mathrm{i}p(x-y)}\,\mathrm{d}p \\ &= \frac{1}{2\pi}\int F\left(\frac{x+y}{2},p\right)\mathrm{e}^{-\mathrm{i}p(x-y)}\,\mathrm{d}p\end{aligned} \tag{161}$$

because F is real. Defining $f(x,y) = (Z^\dagger F)(x,y)$ as in (82), we have

$$\overline{f(x,y)} = \frac{1}{2\pi}\int F\left(\frac{x+y}{2},p\right)\mathrm{e}^{-\mathrm{i}p(x-y)}\,\mathrm{d}p \tag{162}$$

and so

$$\left(\left(\Pi_{L_2(\mathbb{C},\mathrm{d}x)}(g)\otimes\Pi_{L_2(\mathbb{C},\mathrm{d}y)}(g)^C\right)f\right)(x,y) = \left(Z^\dagger\Pi_{\mathcal{K}_R}(g)F\right)(x,y) = \overline{f(x,y)}. \tag{163}$$

Now it can be seen that a possible factorization has

$$\left(\Pi_{L_2(\mathbb{C},\mathrm{d}x)}(g)u\right)(x) = \mathrm{e}^{\mathrm{i}\omega}\overline{u}(x) \qquad \left(\Pi^C_{L_2(\mathbb{C},\mathrm{d}y)}(g)u\right)(x) = \mathrm{e}^{-\mathrm{i}\omega}\overline{v}(y) \tag{164}$$

with ω any real number, so that

$$\Pi_{\mathcal{H}}(g) = \Pi^C_{\mathcal{H}}(g) = \mathrm{e}^{\mathrm{i}\omega}\hat{C} \tag{165}$$

where $\hat{C}$ is the antiunitary operator of (50) and (55).

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